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ON THE I-CENTERS OF A TRIANGLE.

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Introduction. The first two sections below are known. The following two sections are a natural extension of the first two. I believe them to be new. The sections 5, 6, 7, are consequences of the first four applied to the triangle. The rest of the article is an application of the first four sections to the inscribed quadrilateral. The results are undoubtedly new.

Definition. The I-centers of a triangle are the centers of the four circles touching the sides of the triangle.

1. Let ABC (Fig. 1) be a given triangle, I its incenter, I_1 its excenter relative to the vertex A, *i. e.*, the center of the escribed circle touching the side BC, opposite the vertex A, not produced. The lines BI, BI_1 being the bisectors of two adjacent supplementary angles, are perpendicular to each other, hence the segment II_1 subtends a right angle at the point B. The same is true about the point C, for similar reasons. Hence: *The incenter of a triangle and its excenter relative to a given vertex, are the extremities of a diameter of a circle passing through the other two vertices of the triangle.*

2. The center of the circle $BICI_1$ (1) is the point common to II_1 and the perpendicular bisector of BC. Let E, E' be the points where this perpendicular bisector meets the circumcircle (O) of ABC, the points E and A being on opposite sides of BC. Now the line II_1 being the interior bisector of the angle BAC, passes through the mid-point E of the arc BEC of the circumcircle (O), hence: *The center of the circle $BICI_1$ is the mid-point E of the arc of the circumcircle of ABC which is subtended by the side BC and which does not contain the vertex A.*[1]

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[1] Julius Petersen, *Méthodes et théories pour la résolution des problèmes de constructions géométriques*, p. 7. Paris, Gauthier-Villars, 1901, third edition. Clement V. Durell, *A Course of Plane Geometry for Advanced Students*, Part I, p. 33, London, Macmillan and Co., 1909.

3. Let I_2, I_3 (Fig. 1) be the excenters of ABC relative to the vertices B, C. The lines BI_2, BI_3 being the bisectors of two adjacent supplementary angles, are perpendicular to each other, hence the segment I_2I_3 subtends a right angle at the point B. The same is true about the point C, for similar reasons. Hence: *The two excenters of a triangle relative to two given vertices are the extremities of a diameter of a circle passing through the two considered vertices of the triangle.*

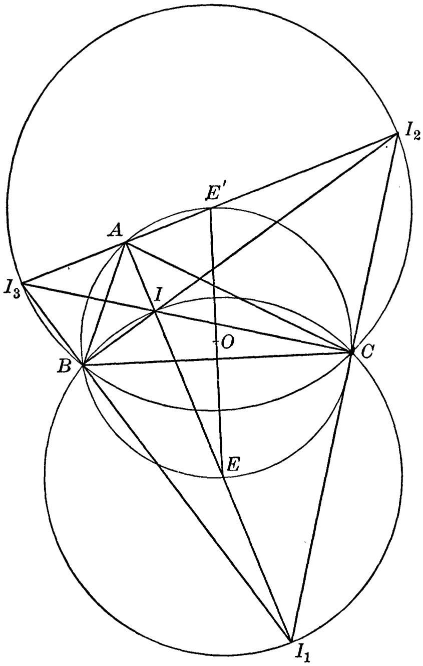

Fig. 1.

4. The center of the circle BI_2CI_3 (3) is the point of intersection of I_2I_3 with the perpendicular bisector EE' of BC. Now the line I_2I_3 being the exterior bisector of the angle BAC, passes through the mid-point E' of the arc $BE'C$ of the circumcircle (O). Hence: *The center of the circle BI_2CI_3 is the mid-point of the arc of the circumcircle of ABC which is subtended by BC and which contains the third vertex of the triangle.*

5. Let FF', GG' be the diameters of the circumcircle (O) of ABC perpendicular to the sides CA, AB respectively, the points F, B lying on opposite sides of CA, and G, C—on opposite sides of AB. The points E, E', F, F', G, G' are the centers of six circles (E), (E'), (F), (F'), (G), (G'), which pass through pairs of vertices of the triangle and through pairs of its I-centers I, I_1, I_2, I_3. We thus obtain the following table (2, 4):

Table I.

(E)	I, I_1	I_2, I_3	(E')
(F)	I, I_2	I_3, I_1	(F')
(G)	I, I_3	I_1, I_2	(G')

Hence: *Each I-center of a triangle lies on three of the six circles which pass through the pairs of vertices of the triangle and have for their centers the mid-points of the arcs subtended by the respective sides of the triangle on its circumcircle.*

Each of the six circles contains two I-centers, and the two points are the extremities of a diameter.

6. The six circles (5) may be used for the construction of the I-centers of the triangle. Therefore: *If the mid-points of the six arcs subtended by the sides of a triangle on its circumcircle are given, the I-centers of the triangle may be constructed by use of the compass alone.*

7. The two circles (E), (E') passing through B, C and having for their respective centers E, E' (2, 4), are independent of the position which the vertex A occupies on the circumcircle (O) of the triangle ABC. Consequently: *If a variable triangle has a fixed base and a fixed circumcircle, its four I-centers describe two circles, each passing through the two fixed vertices and having for their respective centers the extremities of the diameter of the circumcircle which is perpendicular to the fixed side.*

8. Let $M_{2,3}$, $M_{2,3}^{1,4}$ (Fig. 2) denote the mid-points of the two arcs of the circle (C) subtended by the side BC of the quadrilateral $ABCD$ inscribed in (C), the point $M_{2,3}^{1,4}$ being the mid-point of the arc which contains the points A, D. A similar notation will be adopted for the mid-points of the arcs into which (C) is divided by each of the other three sides and by each of the two diagonals of the quadrilateral. Thus $M_{1,3}^{2}$, for instance, will denote the mid-point of the arc subtended by the diagonal AC and which contains the point B. The figure thus involves twelve such mid-points M.

The four points A, B, C, D taken three at a time, determine four triangles, each of which has four I-centers, so that we have in all sixteen I-centers. The incenters of the four triangles BCD, CDA, DAB, ABC, will be denoted by P, Q, R, S, and the excenters of the triangle BCD relative to the vertices B, C, D, by P_2, P_3, P_4, respectively. Similarly for the excenters of the other three triangles.

The I-centers P, P_4, of the triangle BCD are the points of intersection of the circle $(M_{2,3})$ having for its center the point $M_{2,3}$ and passing through B, C (2) with the line $DM_{2,3}$, while the I-centers P_2, P_3 are the points of intersection of the line $DM_{2,3}^{1,4}$ with the circle $(M_{2,3}^{1,4})$ (4). Similarly for the I-centers of the other triangles. The circle $(M_{2,3})$ contains the I-centers P, P_4, of the triangle BCD, and the I-centers S, S_1 of the triangle ABC. Similarly for the other eleven (M) circles. Thus we obtain the following table, where the (M) circle and the I-centers lying on it are written in the same horizontal line, on the same side of the vertical line.

TABLE II.

$(M_{2,3})$	S, S_1, P, P_4	S_2, S_3, P_2, P_3	$(M_{2,3}^{1,4})$
$(M_{1,4})$	R, R_2, Q, Q_3	Q_1, Q_4, R_1, R_4	$(M_{1,4}^{2,3})$
$(M_{1,2})$	R, R_4, S, S_3	R_1, R_2, S_1, S_2	$(M_{1,2}^{3,4})$
$(M_{3,4})$	P, P_2, Q, Q_1	P_3, P_4, Q_3, Q_4	$(M_{3,4}^{1,2})$
$(M_{1,3}^{4})$	Q_1, Q_3, S, S_2	Q, Q_4, S_1, S_3	$(M_{1,3}^{2})$
$(M_{2,4}^{3})$	P_2, P_4, R, R_1	P, P_3, R_2, R_4	$(M_{2,4}^{1})$

Hence: *The sixteen I-centers of the four triangles obtained by taking the vertices of an inscribed quadrilateral three at a time, lie on twelve circles. Each I-center lies on three circles. Each circle contains four I-centers. The centers of those twelve circles all lie on the circumcircle of the quadrilateral.*

9. The two pairs of I-centers lying on the same (M) circle (8) are the extremities of two diameters of this circle (2, 4). Therefore: *The four I-centers lying on the same circle form a rectangle.*

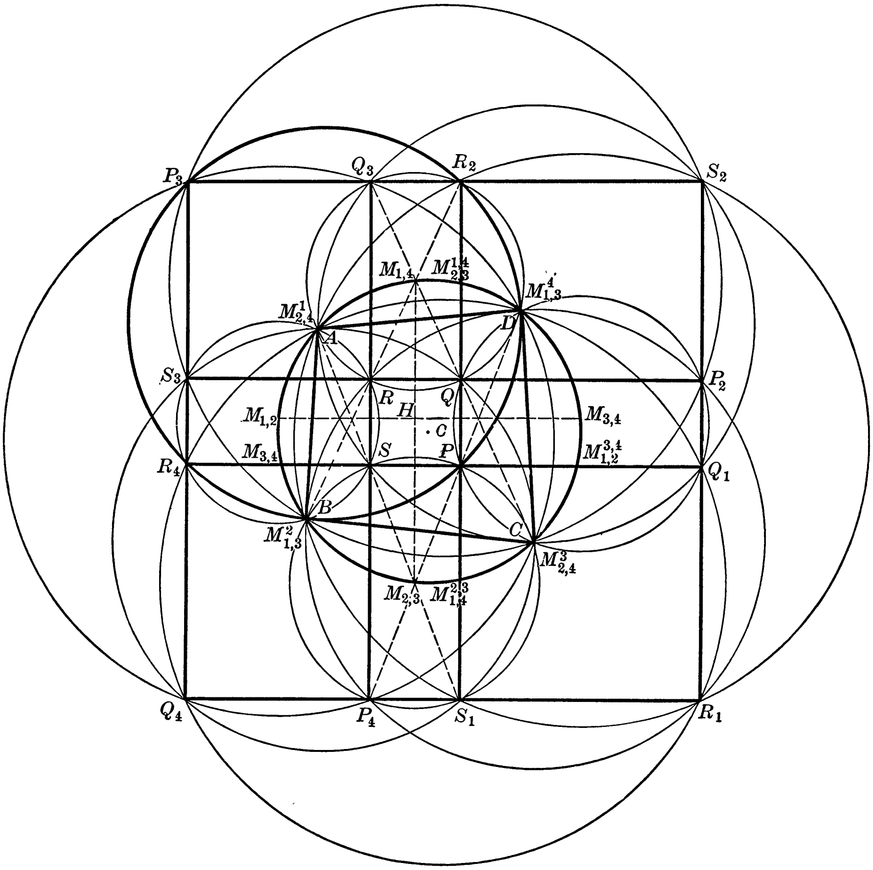

FIG. 2.

10. Let H be the point of intersection of the line $M_{1,2}M_{3,4}$ with the line $M_{2,3}M_{1,4}$. The angle $M_{1,2}HM_{2,3}$ is measured by one half of the sum of the arcs $M_{1,2}BM_{2,3}$ and $M_{3,4}DM_{1,4}$. Now

$$\smile M_{1,2}BM_{2,3} = \smile M_{1,2}B + \smile BM_{2,3} = \tfrac{1}{2} \smile AB + \tfrac{1}{2} \smile BC$$

and

$$\smile M_{3,4}DM_{1,4} = \smile M_{3,4}D + \smile DM_{1,4} = \tfrac{1}{2} \smile CD + \tfrac{1}{2} \smile DA.$$

Hence the angle $M_{1,2}HM_{2,3}$ has for its measure

$$\tfrac{1}{4}\cdot(\smile AB + \smile BC + \smile CD + \smile DA) = \tfrac{1}{4}\cdot 360^\circ = 90^\circ.$$

Therefore: *The lines* $M_{1,2}M_{3,4}$, $M_{2,3}M_{1,4}$ *are perpendicular to each other.*

11. The incenter P of the triangle BCD lies on the same side of BC as the vertex D, and is common to the circle $(M_{2,3})$ (8) and to the bisector $DM_{2,3}$ of the angle BDC inscribed in (C). Similarly the incenter S of the triangle ABC lies on the same side of BC as A, and is common to $(M_{2,3})$ and $AM_{2,3}$. Thus the incenters P, S lie on the same side of BC as the segment AD and are the points of intersection of the circle $(M_{2,3})$ with the lines $M_{2,3}D$, $M_{2,3}A$ respectively.

The line $M_{2,3}M_{1,4}$ is the bisector of the angle $AM_{2,3}D$ inscribed in (C), and therefore also of the angle $SM_{2,3}P \equiv$ angle $AM_{2,3}D$. But the angle $SM_{2,3}P$ is a central angle of the circle $(M_{2,3})$, hence $M_{2,3}M_{1,4}$ is the perpendicular bisector of the chord PS. Similarly the line $M_{1,4}M_{2,3}$ is the perpendicular bisector of the chord QR of the circle $(M_{1,4})$. Thus the two segments PS and QR have the same perpendicular bisector $M_{2,3}M_{1,4}$. By an analogous reasoning it may be shown that the line $M_{1,2}M_{3,4}$ is the common perpendicular bisector of the two segments PQ and RS. The opposite sides of the quadrilateral $PQRS$ are thus perpendicular to the two lines $M_{1,2}M_{3,4}$, $M_{2,3}M_{1,4}$ which are themselves perpendicular to each other. Consequently: *The four incenters of the four triangles determined by the vertices of an inscribed quadrilateral taken three at a time, form a rectangle.*[1]

12. The four points PSP_4S_1 lying on the circle $(M_{2,3})$ (8), form a rectangle (9), hence PS_1 is perpendicular to PS at P. On the other hand $PQRS$ forming a rectangle (11), the perpendicular to PS at P is PQ, hence the three points Q, P, S_1 are collinear. The three points R, S, P_4 are collinear for similar reasons, and the two lines QPS_1 and RSP_4 are parallel, as opposite sides of the rectangle $PQRS$. Considering the rectangles PQP_2Q_1 (8, 9) and $PQRS$ (11), it may be established in a similar way that we have the two parallel lines RQP_2 and SPQ_1.

$RP_4R_1P_2$ and PSP_4S_1 are rectangles inscribed in the circles $(M^3_{2,4})$ and $(M_{2,3})$ respectively (8, 9), hence the lines P_4R_1 and P_4S_1 are the respective perpendiculars to the lines RP_4 and SP_4 at P_4; but the three points R, S, P_4 are collinear (12), hence the lines P_4R_1, P_4S_1 coincide, and the three points P_4, S_1, R_1 are collinear. Considering the two rectangles $RP_4R_1P_2$ and PQP_2Q_1, it may be shown in a similar way that the points P_2, Q_1, R_1 are collinear.

Thus the parallel QPS_1 through P to the side RP_4 of the rectangle $RP_4R_1P_2$ meets the side P_4R_1 of this rectangle in the point S_1, and the parallel SPQ_1 through P to the side RP_2 meets the side P_2R_1 in the point Q_1, hence $PQ_1R_1S_1$ is a rectangle. In an analogous manner it may be shown that we have the rectangles $QP_2S_2R_2$, $RQ_3P_3S_3$, $SP_4Q_4R_4$. Consequently: *If of the four triangles determined by the vertices of an inscribed quadrilateral taken three at a time, the three triangles are taken having a given vertex in common, the three excenters relative to this vertex in the three triangles, are three vertices of a rectangle, the fourth vertex of which is the incenter of the fourth triangle.*

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[1] See this MONTHLY, Vol. XXIV, March, 1917, p. 124.

13. The three rectangles $PQ_1R_1S_1$ (12), PQP_2Q_1 (8, 9), and $QP_2S_2R_2$ (12) show that the points R_1, Q_1, P_2, S_2 are collinear, as well as the points S_1, P, Q, R_2, and the two lines are parallel. Similar considerations may be applied to the three rectangles PSP_4S_1 (8, 9), $PQRS$ (11), QRQ_3R_2 (8, 9), and again to the series $SP_4Q_4R_4$ (12), SRS_3R_4 (8, 9), $RS_3P_3Q_3$ (12). Thus we obtain the four parallel lines $R_1Q_1P_2S_2$, S_1PQR_2, P_4SRQ_3, $Q_4R_4S_3P_3$. Similarly for the four parallel lines $R_1S_1P_4Q_4$, Q_1PSR_4, P_2QRS_3, $S_2R_2Q_3P_3$. Consequently: *The sixteen I-centers of the four triangles determined by the vertices of an inscribed quadrilateral taken three at a time, lie by groups of four on eight straight lines. The eight lines consist of two sets of four parallel lines, and the lines of one set are perpendicular to the lines of the other.*

NOTE ON CONTINUOUS FUNCTIONS.

By CAPTAIN K. P. WILLIAMS.

1. The class of continuous functions is at the same time one of the simplest and most important in analysis. On account of the character of physical phenomena, quantities whose variation is of a continuous sort are naturally the first considered. In the more recently developed topics in analysis other classes of functions are considered, and the idea of continuity does not play such a fundamental rôle. The considerations that follow are intended primarily for those who have just become familiar with the properties of continuous functions as developed from a rigorous point of view. As they are of a simple nature they may serve as an easy introduction to the study of classes of functions where one's starting point is no longer the idea of continuity.

When we examine the properties of continuous functions we find that they fall into two rather broad classifications. There are those properties that have to do with the behavior of the function in the immediate vicinity of a point, and those that relate to some character with reference to the interval of definition. We could call them properties "im kleinen," and properties "im grossen." The definition of continuity itself is an example of the first class of properties, and the theorem that every continuous function has actual extrema in a closed interval, an example of the second. Thus we see how strongly properties of a function "im kleinen" effect its behavior "im grossen." Intimately connected with any property "im kleinen" is the question of uniformity; that is we inquire whether the property in question occurs in a uniform manner throughout the interval of definition. And in an analogous manner we can ask whether a property "im grossen" is true in every sub-interval. With such ideas before us we can examine easily how far various properties of continuous functions are uniquely characteristic.

2. We consider a function $f(x)$ which approaches a definite limit at every point, but which is not necessarily continuous at any point in the interval (a, b). Denote by $f'(x)$ the limit approached. We first prove the